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Classical scattering in Liouville field theory

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Abstract. A detailed description of classical scattering in Liouville field theory (LFT) is presented. Contrary to widespread belief, LFT scattering is shown to be non-trivial, although it is finite-dimensional in some sense. In particular, for certain phase spaces, the LFT S-matrix is represented as a transformation of the Poisson group $SL(2, \mathbb{R})$. The completeness and conformal invariance of the scattering are also indicated. Singular fields are treated on an equal footing with regular ones, except that only the latter are given a consistent Hamiltonian interpretation. A number of unexpected peculiarities of LFT scattering are revealed. First, for some exceptional field configurations, the asymptotic fields are not solutions of d'Alembert's equation, rather they are a sum of the d'Alembert and Liouville components. Second, the scattering occurs in 'two or three spaces'. And last, depending on the choice of the algebra of observables, the conventional splitting of the d'Alembert field into leftward and rightward components is either in general impossible or essentially non-unique.

1. Introduction

It is widely believed that there is no scattering in Liouville field theory (LFT) [1–5]. This belief, however, is in direct contradiction to some of Dzhordzhadze's results on the classical Liouville equation [6]. For convenience of discussion, we shall formulate the relevant part of Dzhordzhadze's paper in the following form.

Proposition [6]. Let $\Phi(t,x) \in C^2(\mathbb{R}^2)$ be a real solution of Liouville's equation:

$$\Box\Phi + 4\,\mathrm{e}^{2\Phi} = 0 \qquad \Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}. \tag{1.1}$$

For this solution, define a tangent field $A_\tau(t,x)$ by the following conditions (τ is a real parameter):

(1) for all $\tau \in \mathbb{R}$, $A_\tau \in C^2(\mathbb{R}^2)$ and $\Box A_\tau = 0$;

(2) $A_\tau(\tau,x) = \Phi(\tau,x)$, $\frac{\partial}{\partial t} A_\tau(t,x)|_{t=\tau} = \frac{\partial}{\partial t}\Phi(t,x)|_{t=\tau}$. (1.2)

Then

(1) for all $(t,x) \in \mathbb{R}^2$ there exist $\lim_{\tau\to\pm\infty} A_\tau(t,x) \equiv A_{\substack{\text{out}\\ \text{in}}}(t,x)$;

(2) $A_{\substack{\text{out}\\ \text{in}}} \in C^2(\mathbb{R}^2)$ and $\Box A_{\substack{\text{out}\\ \text{in}}} = 0$;

(3) $A_{\text{in}} \neq A_{\text{out}}$, i.e. the S-matrix always differs from unity.

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The authors of the paper [3] disagreed with Dzhordzhadze and claimed that the classical LFT S-matrix was trivial, but their argument was indirect, i.e. not based on investigation of the spacetime behaviour of the fundamental field Φ, and hence inconclusive. On the other hand, we have checked Dzhordzhadze's assertions and found them to be perfectly correct. Basing ourselves on our previous work [7], we describe here the LFT classical scattering, as well as its Hamiltonian interpretation, in more detail. The quantum scattering will be discussed in a subsequent paper [13].

We have found the LFT classical scattering to be non-trivial, complete, and conformally invariant. Since the conformal transformations may connect the states of arbitrary energy and momentum, the last property (conformal invariance) implies that the scattering is purely internal, i.e. independent of energy and momentum. This tempts one to conclude wrongly that the S-matrix is unity, the more so as the fundamental field is scalar. One should not fall into this trap. Perhaps the most interesting result is that the LFT scattering is essentially finite-dimensional, and, for some phase spaces, it is possible to represent the S-matrix in a particularly simple form as a transformation of the Poisson group $SL(2,\mathbb{R})$. Also, three curious phenomena are worthy of note. First, for some exceptional field configurations, the asymptotic fields are not solutions of d'Alembert's equation, rather they are a sum of the d'Alembert and Liouville components. Second, the scattering occurs in 'two or three spaces' (see comment 3 below). And last, the conventional splitting of the d'Alembert field into leftward and rightward components is in general impossible if some natural algebra of admissible functionals (observables) is used. Also, after natural extension of the admissible algebra such splitting becomes possible but essentially non-unique. Now some comments on the above proposition.

(1) Note that no restrictions are imposed on behaviour of the fundamental field Φ at spatial infinity. We shall impose some natural boundary conditions, which, among other things, allow the Poisson structure to be introduced [7].

(2) For singular solutions [7–10], the above proposition cannot in general hold true, e.g. the following solution

$$\Phi(t,x) = -\log\frac{2|x-vt-q|}{\sqrt{1-v^2}} \qquad q, v \in \mathbb{R},\ |v|<1 \tag{1.3}$$

exhibits no scattering. This is rather an exceptional behaviour. We shall treat the singular solutions on an equal footing with the regular ones, except that only the latter will be given a consistent Hamiltonian interpretation. Due to our choice of the phase spaces of the model, the singular solutions always exhibit non-trivial scattering. Solution (1.3) is not a counter-example, because it does not belong to any of the adopted phase spaces. Nevertheless, this solution will emerge later on, but as a part of the asymptotic field for some exceptional field configurations.

(3) The asymptotic fields are also uniquely fixed in the present context by the following limiting relation, which we shall exploit in the main body of the paper:

$$\forall x, v \in \mathbb{R}, |v| \leqslant 1\ \exists \lim_{t\to\pm\infty} |\,\Phi(t,x+vt) - A_{\substack{\mathrm{out}\\ \mathrm{in}}}(t,x+vt)| = 0. \tag{1.4}$$

This asymptotic condition permits avoiding the introduction of the tangent field A_τ. The reason for A_τ being unwanted is as follows: the boundary conditions and singularities which are appropriate for the Cauchy data for d'Alembert's equation differ from those appropriate for the Cauchy data in LFT, so the equalities (1.2) appear to be inconsistent. One can interpret this situation in terms of the scattering in 'two spaces'. Moreover, for some phase spaces, the set of in-fields does not coincide with the set of out-fields, i.e. for these LFT phase spaces, scattering occurs in 'three spaces', in which case it is impossible for the S-matrix to be represented in the abovementioned simple form.

2. Asymptotic fields and the S-matrix

We begin with the description of the field configurations for which scattering will be studied. The details can be found in [7]. Let $\varphi(x)$ and $\pi(x)$ be the Cauchy data corresponding to the solution $\Phi(t,x)$, i.e.

$$\varphi(x) = \Phi(0,x) \qquad \pi(x) = \left.\frac{\partial}{\partial t}\Phi(t,x)\right|_{t=0}.$$

It is easier to describe all the conditions imposed on φ and π in terms of two potentials

$$U_\pm(x) = \left(\frac{\varphi'(x) \pm \pi(x)}{2}\right)^2 - \left(\frac{\varphi'(x) \pm \pi(x)}{2}\right)' + \mathrm{e}^{2\varphi(x)}. \tag{2.1}$$

First, it is required that $U_\pm \in S(\mathbb{R})$, where $S(\mathbb{R})$ is a real Schwartz space consisting of real C^∞-smooth functions of $x \in \mathbb{R}$ that fall off with all their derivatives more rapidly than any power of x^{-1} as $|x| \to \infty$. Second, both potentials $U_\pm$ must not possess a virtual eigenvalue. To make these requirements more exact and facilitate further considerations it is convenient to introduce some notations.

Schrödinger's equation $-f''(x) + U(x)f(x) = 0$ with $U \in S(\mathbb{R})$ possesses solutions exhibiting the following asymptotic behaviour:

$$\begin{aligned} \psi_1(x) &= 1 + s_-(x) = \alpha + \beta x + s_+(x) \\ \chi_1(x) &= \vartheta - \beta x + s_-(x) = 1 + s_+(x) \end{aligned} \tag{2.2}$$

where α, β and ϑ are some x-independent constants (depending on U), and a notation similar to $o(x)$ is used: let $\eta(x)$ be a real C^∞ function such that $\eta(x) = 0$ if $x < 0$, and $\eta(x) = 1$ if $x > 1$, then for $f \in C^\infty(\mathbb{R})$, the equality $f(x) = s_+(x)$ means that $\eta f \in S(\mathbb{R})$, and $f(x) = s_-(x)$ means that $f(-x) = s_+(x)$. For example, $\tanh x = 1 + s_+(x) = -1 + s_-(x)$.

Define some subsets of $S(\mathbb{R})$ ($n = 0, 1, 2, \ldots$):

$$M_n = \left\{ U \,\middle|\, \begin{array}{l} U \in S(\mathbb{R}),\ \text{the solution } \psi_1 \text{ corresponding to } U \text{ is} \\ \text{unbounded, i.e. } \beta \neq 0, \text{ and has exactly } n \text{ zeros} \end{array} \right\}. \tag{2.3}$$

For the potential $U \in \bigcup_{n=0}^{\infty} M_n$, one can define another two solutions of Schrödinger's equation by the formulae

$$\psi(x) = |\beta|^{-1/2}\psi_1(x) \qquad \chi(x) = |\beta|^{-1/2}\chi_1(x). \tag{2.4}$$

Consider a set

$$\mathcal{M} = \bigcup_{\substack{n_+=0 \\ n_-=0}}^{\infty} M_{n_+} \times M_{n_-} \times PSL(2,\mathbb{R}) \tag{2.5}$$

where $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\mathrm{id}\}$. $\mathcal{M}$ is an open and disconnected subset of $S(\mathbb{R}) \times S(\mathbb{R}) \times PSL(2,\mathbb{R})$, the triples (U_+, U_-, T) being its points (the quantities introduced above and corresponding to the potentials $U_\pm$ will now acquire the '$\pm$' indices). Now define an injective mapping from $\mathcal{M}$ into field configurations

$$J : (U_+, U_-, T) \to (\pi, \varphi) \tag{2.6}$$

as follows. If U_+, U_-, and T are given ($U_\pm \in M_{n_\pm}$), then

$$\varphi(x) = -\log|\Omega_+(x)T\Omega_-(x)|$$
$$\pi(x) = -\frac{\Omega'_+(x)T\Omega_-(x) - \Omega_+(x)T\Omega'_-(x)}{\Omega_+(x)T\Omega_-(x)} \tag{2.7}$$

where

$$\Omega_+(x) = \left(\chi_+(x), (-1)^{n_+}\psi_+(x)\right)$$
$$\Omega_-(x) = \begin{pmatrix} (-1)^{n_-}\psi_-(x) \\ \chi_-(x) \end{pmatrix} \tag{2.8}$$

and the element $T \in PSL(2,\mathbb{R})$ is regarded as a 2×2 real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$. The scattering will be studied for those field configurations which constitute the image $\mathcal{F}$ of the mapping J.

The Cauchy data (2.7) generate a solution to the equation (1.1) of the form

$$\Phi(t,x) = -\log|\Omega_+(x^+)T\Omega_-(x^-)| \tag{2.9}$$

where the cone variables $x^\pm = x \pm t$ are used. Note also that the solution (1.3) corresponds to the potentials $U_\pm(x) \equiv 0$, which have virtual eigenvalues, and hence, this field configuration is not allowed.

Now we are in a position to introduce asymptotic fields. For given $U_\pm \in M_{n_\pm}$ and $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, define

$$A_{\rm in}(t,x) = \begin{cases} -\log\left|\Omega_+(x^+)T\begin{pmatrix}1\\0\end{pmatrix}a^{-1}(1,0)T\Omega_-(x^-)\right| & \text{if } a \neq 0 \\ -\log|\psi_{+1}(x^+)\chi_{-1}(x^-)| - \log\dfrac{2|x - v_{\rm in}t - q_{\rm in}|}{\sqrt{1 - v_{\rm in}^2}} & \text{if } a = 0 \end{cases} \tag{2.10}$$

$$A_{\rm out}(t,x) = \begin{cases} -\log\left|\Omega_+(x^+)T\begin{pmatrix}0\\1\end{pmatrix}d^{-1}(0,1)T\Omega_-(x^-)\right| & \text{if } d \neq 0 \\ -\log|\chi_{+1}(x^+)\psi_{-1}(x^-)| - \log\dfrac{2|x - v_{\rm out}t - q_{\rm out}|}{\sqrt{1 - v_{\rm out}^2}} & \text{if } d = 0 \end{cases} \tag{2.11}$$

where

$$v_{\rm in} = \frac{|\beta_-| - b^2|\beta_+|}{|\beta_-| + b^2|\beta_+|} \qquad q_{\rm in} = \frac{(-1)^{n_+}\vartheta_+ b^2 - (-1)^{n_-}\alpha_- + bd}{|\beta_-| + b^2|\beta_+|} \tag{2.12}$$

$$v_{\rm out} = \frac{b^2|\beta_-| - |\beta_+|}{b^2|\beta_-| + |\beta_+|} \qquad q_{\rm out} = \frac{(-1)^{n_-}\vartheta_- b^2 - (-1)^{n_+}\alpha_+ + ab}{b^2|\beta_-| + |\beta_+|}. \tag{2.13}$$

Note that if $a = 0$, then $A_{\rm in}$ does not solve d'Alembert's equation because of the term of Liouville type (1.3). The same is true for $A_{\rm out}$ when $d = 0$.

Proposition. (1) If Φ, $A_{\rm in}$, and $A_{\rm out}$ are given by (2.8)–(2.13), then the asymptotic condition (1.4) is satisfied for all $x, v \in \mathbb{R}$, $|v| \leqslant 1$, except for a finite number of pairs (x, v) that correspond to the straight lines of singularities of $A_{\rm in}$ and $A_{\rm out}$.

(2) The asymptotic condition (1.4) defines uniquely the d'Alembert components of $A_{\rm in}$ and A_{out}, as well as the parameters v_{in}, q_{in}, v_{out}, and q_{out}, which specifies the Liouville components of $A_{\rm in}$ and $A_{\rm out}$.

The proof is a mere use of the asymptotics (2.2) and definitions (2.4), (2.8)–(2.13).

The singular lines of $A_{\rm in}$ are nothing but asymptotes for the singular curves of Φ when $t \to -\infty$. Likewise the singular lines of $A_{\rm out}$ serve as asymptotes for the same curves when $t \to +\infty$. Explicit description of the boundary behaviour and singularities of the asymptotic fields may be found in appendices A and B.

Formulae (2.10)–(2.13) define two mappings: from $\mathcal{M}$ into the set of in-fields, and from $\mathcal{M}$ into the set of out-fields,

$$J_\sharp : (U_+, U_-, T) \to (\pi_\sharp, \varphi_\sharp). \tag{2.14}$$

Here and below $\sharp$ stands for either 'in' or 'out', and the zero-time fields

$$\varphi_\sharp(x) = A_\sharp(0, x) \qquad \pi_\sharp(x) = \frac{\partial}{\partial t} A_\sharp(t, x)\bigg|_{t=0}$$

are introduced. Both the mappings are injective. Indeed, for given $\pi_\sharp$ and $\varphi_\sharp$, the potentials $U_\pm$ can be obtained as follows:

$$U_\pm(x) = \left(\frac{\varphi_\sharp'(x) \pm \pi_\sharp(x)}{2}\right)^2 - \left(\frac{\varphi_\sharp'(x) \pm \pi_\sharp(x)}{2}\right)' \tag{2.15}$$

(if the Liouville component is present, it must be dropped), then the matrix T can be extracted from the formulae describing the boundary behaviour of $\pi_\sharp$ and $\varphi_\sharp$ (see appendix A).

Denote the image of $J_\sharp$ as $\mathcal{F}_\sharp$. For the reasons mentioned in section 1, $\mathcal{F} \cap \mathcal{F}_\sharp = \emptyset$, i.e. scattering occurs in more than one space. Further, although the types of possible singularities of in-fields and out-fields are the same, their boundary behaviour may be of different types (see appendix A), i.e. $\mathcal{F}_{\rm in} \neq \mathcal{F}_{\rm out}$. The intersection $\mathcal{F}_{\rm ex} = \mathcal{F}_{\rm in} \cap \mathcal{F}_{\rm out}$ can be described as follows:

$$J_\sharp^{-1}(\mathcal{F}_{\rm ex}) = \{(U_+, U_-, T) \in \mathcal{M} \mid abcd \neq 0\} \equiv \mathcal{M}_0 \,. \tag{2.16}$$

Thus, for the fields in $J(\mathcal{M}_0)$, the scattering occurs in two spaces, otherwise in three.

We may now introduce the wave operators and S-matrix:

$$W_\sharp = J \circ J_\sharp^{-1} : \mathcal{F}_\sharp \to \mathcal{F} \tag{2.17}$$

$$S = W_{\rm out}^{-1} \circ W_{\rm in} = J_{\rm out} \circ J_{\rm in}^{-1} : \mathcal{F}_{\rm in} \to \mathcal{F}_{\rm out} \,. \tag{2.18}$$

All these mappings are bijective, hence the scattering is *complete*. To discuss other properties of these mappings let us designate as E_0^t, E^t and $E_\sharp^t$ the time evolutions in the sets $\mathcal{M}$, $\mathcal{F}$

and $\mathcal{F}_\sharp$,, i.e.

$$E_0^t(U_+(x), U_-(x), T) = (U_+(x+t), U_-(x-t), T)$$

$$E^t(\pi(x), \varphi(x)) = \left(\frac{\partial}{\partial t}\Phi(t,x), \Phi(t,x)\right) \tag{2.19}$$

$$E_\sharp^t(\pi_\sharp(x), \varphi_\sharp(x)) = \left(\frac{\partial}{\partial t}A_\sharp(t,x), A_\sharp(t,x)\right).$$

By definition, they are related as follows:

$$E^t \circ J = J \circ E_0^t \qquad E_\sharp^t \circ J_\sharp = J_\sharp \circ E_0^t. \tag{2.20}$$

This, together with the definitions (2.17), (2.18), leads to the standard inter-twining property of the wave operators and the S-matrix:

$$E^t \circ W_\sharp = W_\sharp \circ E_\sharp^t \qquad E_{\mathrm{out}}^t \circ S = S \circ E_{\mathrm{in}}^t. \tag{2.21}$$

The insiders' favourite way of introducing wave operators is to use the limit of the type

$$W_{\mathrm{in}} = \lim_{t\to-\infty} E^{-t} \circ I \circ E_{\mathrm{in}}^t \tag{2.22}$$

where $I : \mathcal{F}_{\mathrm{in}} \to \mathcal{F}$ is some identifying mapping, more or less natural. The most direct choice for I is W_{in} itself. In this case the limit (2.22) becomes trivial (and useless) by virtue of (2.21). We suspect that there is no other natural or useful choice for I.

The time evolution is only a small part of the conformal group of the two-dimensional Minkowski space $\mathbb{M}^2$, which is a symmetry group of Liouville's and d'Alembert's equations. In [7] we used a maximal conformal group $\mathfrak{G} = \mathfrak{D} \times \mathfrak{D}$ that consists of the mappings $F : \mathbb{M}^2 \to \mathbb{M}^2$ of the form

$$x^+ \to y^+ = F^+(x^+) \qquad x^- \to y^- = F^-(x^-)$$

$$F^\pm \in \mathfrak{D} = \{G | G \in \mathrm{Diff}_+^\infty(\mathbb{R}), G'' \in S(\mathbb{R})\}$$

where the cone coordinates on $\mathbb{M}^2$ are used, and $\mathrm{Diff}_+^\infty(\mathbb{R})$ designates the group of orientation-preserving C^∞ diffeomorphisms of $\mathbb{R}$. The intertwining properties (2.20), (2.21) can be generalized to include conformal transformations. Namely, for $F = F^+ \times F_+ \in \mathfrak{G}$, we have

$$E^F \circ W_\sharp = W_\sharp \circ E_\sharp^F \qquad E_{\mathrm{out}}^F \circ S = S \circ E_{\mathrm{in}}^F. \tag{2.23}$$

These are a consequence of

$$E^F \circ J = J \circ E_0^F \qquad E_\sharp^F \circ J_\sharp = J_\sharp \circ E_0^F.$$

The F-evolutions are defined as follows:

$$E_0^F(U_+(x), U_-(x), T) = (U_+^F(x), U_-^F(x), T^F)$$

$$E^F(\pi(x), \varphi(x)) = \left(\frac{\partial}{\partial t}\Phi^F(t,x)\Big|_{t=0}, \Phi^F(0,x)\right)$$

$$E_\sharp^F(\pi_\sharp(x), \varphi_\sharp(x)) = \left(\frac{\partial}{\partial t}A_\sharp^F(t,x)\Big|_{t=0}, A_\sharp^F(0,x)\right)$$

where [7]

$$U_{\pm}^{F}(x) = U_{\pm}(F^{\pm}(x))(\partial F^{\pm}(x))^2 + \left(\frac{\partial^2 F^{\pm}(x)}{2\partial F^{\pm}(x)}\right)^2 - \left(\frac{\partial^2 F^{\pm}(x)}{2\partial F^{\pm}(x)}\right)'$$

$$T^F = \begin{pmatrix} \frac{1}{r_+} & 0 \\ 0 & r_+ \end{pmatrix} T \begin{pmatrix} \frac{1}{r_-} & 0 \\ 0 & r_- \end{pmatrix}^{-1} \qquad r_{\pm} = \left(\frac{\partial F^{\pm}(+)}{\partial F^{\pm}(-)}\right)^{1/4}$$

(here an abbreviation of the kind $\partial G(\pm) = \lim_{t\to\pm\infty} \partial G(x)$ is used), and in the cone variables

$$\Phi^F(x^+, x^-) = \Phi(F^+(x^+), F^-(x^-)) + \tfrac{1}{2}\log(\partial F^+(x^+)\partial F^-(x^-)).$$

The asymptotic fields $A_{\natural}$ are transformed in the same way as Φ when the Liouville component is absent. The response of $A_{\natural}$ to the conformal transformation in the presence of the Liouville component is described in appendix C.

High symmetry of the S-matrix (2.23) suggests that it must be simple in some sense. However, it is rather senseless to speak about the simplicity of a mapping that connects different sets (e.g. $W_{\natural}$). So let us restrict the S-matrix to the set $\mathcal{F}_{\mathrm{ex}}$ (2.16) to obtain an auto-transformation. Then the mapping $S' = J_{\mathrm{in}}^{-1} \circ S \circ J_{\mathrm{in}} = J_{\mathrm{in}}^{-1} \circ J_{\mathrm{out}} : \mathcal{M}_0 \to \mathcal{M}_0$ appears to have a remarkably simple form

$$S'(U_+, U_-, T) = (U_+, U_-, S'(T)) \tag{2.24}$$

where

$$S'(T) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$
$$b' = b \qquad c' = c \qquad a' = \frac{bc}{1+bc}a \qquad d' = \frac{1+bc}{bc}d. \tag{2.25}$$

This is the very representation of the S-matrix as a transformation of the group $SL(2,\mathbb{R})$ promised in section 1. It is easily seen from (2.24), (2.25) that S' has no fixed points, so the full S-matrix (2.18) is *non-trivial* for any field configuration.

It is not difficult to extend the above constructions to cover the field configurations corresponding to the potentials $U_{\pm}$ possessing the virtual eigenvalues. The reason for our excluding such potentials is that in this case we are able to embed the above results in the context of Hamiltonian theory.

3. Hamiltonian interpretation

In this section we show that the LFT Poisson structure [7] may be considered at the same time as a Poisson structure of d'Alembert's equation, so that the asymptotic fields are locally commutative and canonical. An essential constituent of the LFT Poisson structure is the second KdV Poisson structure. Let us begin with the description of the latter.

For $V \in S(\mathbb{R})$, let

$$\mathrm{d}_V f = \int \mathrm{d}x\, V(x) \frac{\delta f}{\delta U(x)}$$

be a directional derivative of a C^1-class functional $f : M \to \mathbb{R}$, $M \subset S(\mathbb{R})$. Introduce also two more differential operators

$$\mathrm{d}_\pm f = \lim_{x\to\pm\infty} \frac{\partial}{\partial x}\frac{\delta f}{\delta U(x)}$$

in those cases where this limit makes sense and exists. For example, this is the case if $(\partial^2/\partial x^2)(\delta f/\delta U(x)) \in S(\mathbb{R})$ as a function of x.

Let f be a real functional defined on an open set $M \subset S(\mathbb{R})$. We write $f \in \mathcal{O}'(M)$ if the following recursive conditions are fulfilled:

(1) f is a C^1 functional with respect to the variable $U \in M$; $(\partial^2/\partial x^2)(\delta f/\delta U(x)) \in S(\mathbb{R})$ as a function of x, and condition 2 below is satisfied;

(2) the functionals $\mathrm{d}_+ f$, $\mathrm{d}_- f$ and $\mathrm{d}_V f$, $V \in S(\mathbb{R})$ satisfy condition 1 above, $(\mathrm{d}_+\mathrm{d}_- - \mathrm{d}_-\mathrm{d}_+)f = 0$ and for all $V \in S(\mathbb{R})$ $(\mathrm{d}_\pm \mathrm{d}_V - \mathrm{d}_V\mathrm{d}_\pm)f = 0$.

If the equality $(\mathrm{d}_+\mathrm{d}_- - \mathrm{d}_-\mathrm{d}_+)f = 0$ in condition 2 of this definition is replaced by $(\mathrm{d}_+ - \mathrm{d}_-)f = 0$, then we obtain a definition of $\mathcal{O}''(M) \subset \mathcal{O}'(M)$.

The KdV Poisson bracket is defined on the algebras of admissible functionals $\mathcal{O}'(M)$ and $\mathcal{O}''(M)$ as follows:

$$\{f, g\} = (f, g) + \langle f, g\rangle + \tfrac{1}{4}\langle\langle f, g\rangle\rangle \tag{3.1}$$

where

$$(f, g) = -\int \mathrm{d}x\, U(x)\left[\frac{\delta f}{\delta U(x)}\left(\frac{\delta g}{\delta U(x)}\right)' - \left(\frac{\delta f}{\delta U(x)}\right)'\frac{\delta g}{\delta U(x)}\right]$$

$$\langle f, g\rangle = \tfrac{1}{4}\int \mathrm{d}x\left[\left(\frac{\delta f}{\delta U(x)}\right)''\left(\frac{\delta g}{\delta U(x)}\right)' - \left(\frac{\delta f}{\delta U(x)}\right)'\left(\frac{\delta g}{\delta U(x)}\right)''\right]$$

$$\langle\langle f, g\rangle\rangle = \mathrm{d}_+ f\, \mathrm{d}_- g - \mathrm{d}_- f\, \mathrm{d}_+ g.$$

The last term in (3.1) vanishes on the algebra $\mathcal{O}''(M)$.

The functional $f(U_+, U_-, T)$ defined on the space $\mathcal{M}$ (2.5) is said to be admissible in the sense $\mathcal{O}' \times \mathcal{O}'$ (or $\mathcal{O}'' \times \mathcal{O}''$) if it is $\mathcal{O}'$-admissible ($\mathcal{O}''$-admissible) with respect to the first and second arguments, and various 'partial' derivatives exist and commute. The overall Poisson bracket is defined as follows:

$$\begin{aligned}\{f, g\} = \{f, g\}_+ - \{f, g\}_- &+ \mathrm{tr}\left(\frac{\partial f}{\partial T}\otimes\frac{\partial g}{\partial T}\{T \overset{\otimes}{,} T\}\right)\\ &+ \mathrm{tr}\,(\hat f \otimes \hat g \Pi) + \mathrm{tr}\ \Xi\left(\hat f \otimes \frac{\partial g}{\partial T} - \hat g \otimes \frac{\partial f}{\partial T}\right)\end{aligned} \tag{3.2}$$

where $\{\cdot, \cdot\}_\pm$ is the KdV Poisson bracket (3.1) with respect to $U_\pm$,

$$\frac{\partial f}{\partial T} = \begin{pmatrix} \dfrac{\partial f}{\partial a} & \dfrac{\partial f}{\partial c} \\[2mm] \dfrac{\partial f}{\partial b} & \dfrac{\partial f}{\partial d}\end{pmatrix} \qquad \text{if } T = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

$$\{T \overset{\otimes}{,} T\} = [r, T \otimes T] \qquad r = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix} \tag{3.3}$$

(the square brackets designate the matrix commutator),

$$\Pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p & u & v & q \\ -p & -v & -u & -q \\ 0 & s & -s & 0 \end{pmatrix} \tag{3.4}$$

$$\Xi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes (h_1 \sigma T + h_2 T \sigma) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (k_1 \sigma T + k_2 T \sigma) \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5}$$

(nine non-dynamical real parameters $p, u, v, q, s, h_1, h_2, k_1, k_2$ appearing in (3.4), (3.5), are specified in [7]), and lastly the 'cap' operation is defined as follows:

$$\hat{f} = \begin{pmatrix} \mathrm{D}^+_+ f & \mathrm{D}^+_- f \\ \mathrm{D}^-_+ f & \mathrm{D}^-_- f \end{pmatrix} \qquad \mathrm{D}^+_\pm = \mathrm{d}^{(+)}_+ \pm \mathrm{d}^{(+)}_- \qquad \mathrm{D}^-_\pm = \mathrm{d}^{(-)}_+ \pm \mathrm{d}^{(-)}_- \tag{3.6}$$

(the upper '$\pm$' correspond to the indices of $U_\pm$). The last two 'trace' terms in (3.2) vanish for the functionals from $\mathcal{O}'' \times \mathcal{O}''$, and hence in this case no parameters participate in the definition of the Poisson bracket. The formula (3.3) alone equips $SL(2, \mathbb{R})$ with the structure of the Poisson Lie group.

The bracket just described is degenerate [7]. To obtain a non-degenerate Poisson structure one should single out from $\mathcal{M}$ a particular phase space, i.e. a minimal set invariant under the Hamiltonian flows. Note that the splitting of $\mathcal{M}$ into phase spaces depends on the choice of the algebra of admissible functionals.

The functionals $\alpha_\pm$ and $\vartheta_\pm$ (2.2) are not admissible; $\beta_\pm$, $\psi_{\pm 1}(x)$, $\chi_{\pm 1}(x)$ are admissible in the sense $\mathcal{O}' \times \mathcal{O}'$; T and $\Omega_\pm(x)$ are admissible in the sense $\mathcal{O}'' \times \mathcal{O}''$ (it is implied that x-dependent quantities should be 'smeared' by the test function from $S(\mathbb{R})$). It is now clear that we must exclude from our consideration the field configurations having the Liouville component, because $q_\sharp$ depends on $\alpha_\pm$ and $\vartheta_\pm$. This is not the whole story. After having a look at (2.10), (2.11), one could expect that $\pi_\sharp$, $\varphi_\sharp$ are $\mathcal{O}'' \times \mathcal{O}''$-admissible. This is not exactly so. When $\pi_\sharp$, $\varphi_\sharp$ have singularities, the variational derivatives of their smeared versions are not smooth, contrary to our definition of admissible functional (the singularities of variational derivatives are of the same type as those of variational derivatives of π, φ, see [7]). An even worse problem with the singular fields is that of non-positiveness of the Hamiltonian [7, 11, 12]. More exactly, the Hamiltonian of the model is positive only on the phase spaces that are subsets of $M_0 \times M_0 \times PSL(2, \mathbb{R})$ [7]. All non-singular fields enter this set and are characterized by an additional condition: $a, d > 0$, $b, c \geqslant 0$. In what follows we assume that only non-singular fields are considered, so that the smeared versions of $\pi_\sharp$, $\varphi_\sharp$ are $\mathcal{O}'' \times \mathcal{O}''$-admissible. However, for the sake of brevity, we shall be omitting the smearing.

Now we are going to demonstrate that the brackets

$$\{A_\sharp(x^+, x^-), A_\sharp(y^+, y^-)\} = \tfrac{1}{4}\,\mathrm{sign}\,(x^+ - y^+) - \tfrac{1}{4}\,\mathrm{sign}\,(x^- - y^-) \tag{3.7}$$

which are appropriate for the d'Alembert fields, are indeed valid. The starting point is the bracket (3.3) together with the following ones [7]:

$$\begin{aligned} \{\Omega_+(x) \overset{\otimes}{,} \Omega_+(y)\} &= \Omega_+(x) \otimes \Omega_+(y) \rho(x - y) \\ \{\Omega_-(x) \overset{\otimes}{,} \Omega_-(y)\} &= -\rho(x - y) \Omega_-(x) \otimes \Omega_-(y) \\ \{\Omega_+(x) \overset{\otimes}{,} \Omega_-(y)\} &= 0 \qquad \{\Omega_\pm(x) \overset{\otimes}{,} T\} = 0 \end{aligned} \tag{3.8}$$

where $\rho(x) = r^{\mathrm{T}}\theta(x) - r\theta(-x)$, $\theta(x)$ being the step function: $\theta(x) = 1$ if $x > 0$, and $\theta(x) + \theta(-x) = 1$. The first two brackets in (3.8) possess a remarkable property of invariance under the action of the Poisson group $SL(2, \mathbb{R})$, i.e.

$$\begin{aligned} \{\Omega_+(x)T \overset{\otimes}{,} \Omega_+(y)T\} &= \Omega_+(x)T \otimes \Omega_+(y)T\rho(x-y) \\ \{T\Omega_-(x) \overset{\otimes}{,} T\Omega_-(y)\} &= -\rho(x-y)T\Omega_-(x) \otimes T\Omega_-(y). \end{aligned} \tag{3.9}$$

For definiteness, we now restrict our consideration to the in-field A_{in}. The out-field is treated in the same way. The column $\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$ and the row $(1,0) \otimes (1,0)$ are eigenvectors of the matrix $\rho(x)$, i.e.

$$\begin{aligned} \rho(x)\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} &= \tfrac{1}{4}\,\mathrm{sign}\,(x)\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \\ (1,0) \otimes (1,0)\rho(x) &= \tfrac{1}{4}\,\mathrm{sign}\,(x)(1,0) \otimes (1,0). \end{aligned}$$

After convolution of these eigenvectors with the row and column (3.9) we obtain

$$\begin{aligned} \left\{\log\left|\Omega_+(x)T\begin{pmatrix}1\\0\end{pmatrix}\right|, \log\left|\Omega_+(y)T\begin{pmatrix}1\\0\end{pmatrix}\right|\right\} &= \tfrac{1}{4}\,\mathrm{sign}\,(x-y) \\ \{\log|(1,0)T\Omega_-(x)|, \log|(1,0)T\Omega_-(y)|\} &= -\tfrac{1}{4}\,\mathrm{sign}\,(x-y). \end{aligned} \tag{3.10}$$

Another bracket we need is

$$\left\{\log\left|\Omega_+(x)T\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{a}\right|, \log\left|\frac{1}{a}(1,0)T\Omega_-(y)\right|\right\} = 0. \tag{3.11}$$

A non-zero result in (3.11) might arise only from the Poisson brackets between the matrix elements of T. Note that the left multiplier depends on T via the combination c/a, whereas the right one via b/a, hence (3.11) is a consequence of the bracket $\{c/a, b/a\} = 0$. After due rearrangement the bracket $\{A_{\mathrm{in}}(x^+, x^-), A_{\mathrm{in}}(y^+, y^-)\}$ amounts to a sum of the brackets (3.10), (3.11), and we obtain (3.7).

The transformation (2.25) preserves the Poisson structure (3.3) on $SL(2, \mathbb{R})$, but it is not defined everywhere in $SL(2, \mathbb{R})$. We are interested, however, in the scattering transformation for particular phase spaces. Let us describe those of them for which the fields are non-singular (i.e. necessarily $U_\pm \in M_0$). Consider first the admissible algebra $\mathcal{O}'' \times \mathcal{O}''$. In this case the phase spaces coincide with the symplectic leaves of (3.2), which in turn are completely determined by the symplectic leaves of $PSL(2, \mathbb{R})$.

(1) $b > 0$, $a > 0$, $c = \varkappa b$ (the non-dynamical constant $\varkappa > 0$ is a label of the leaf);

(2) $b > 0$, $c = 0$, $a > 0$;

(3) $b = 0$, $c > 0$, $a > 0$;

(4) $b = c = 0$, $a =$ const > 0. This is a zero-dimensional leaf of $PSL(2, \mathbb{R})$ labelled by a, which may be here regarded as non-dynamical (the leaves (1)–(3) are two-dimensional).

Let, now, the admissible algebra be $\mathcal{O}' \times \mathcal{O}'$. In this case the symplectic leaves of (3.2) are no longer determined by the leaves of $PSL(2, \mathbb{R})$. Moreover, in [7] different sets of parameters that appear in (3.4), (3.5), and come now into action, were used for different leaves to obtain phase spaces that admit the Hamiltonian representation of the conformal algebra $\mathcal{A}$ (equal to the Lie algebra of the conformal group $\mathfrak{G}$). We assume that the set of parameters have been chosen for a particular phase space in the same way as in [7]. Thus these phase spaces cannot be considererd as symplectic leaves of some overall Poisson structure. The following are the phase spaces corresponding to the non-singular fields.

(1) $b > 0$, $c > 0$, $a > 0$;

(2) $b > 0$, $c = 0$, $a > 0$;

(3) $b = 0$, $c > 0$, $a > 0$;

(4) $b = c = 0$, $a > 0$ (note that a is here a dynamical variable, not just a label, as in the analogous item above).

The scattering transformation (2.25) is a smooth symplectic auto-transformation of the phase spaces of type (1). For the spaces (2)–(4), the S-matrix cannot be represented in the simple form (2.24); and for the other spaces, fundamental fields are singular.

4. Left- and right-movers

The conventional way of studying the d'Alembert field $A(t, x)$ is to split it into the left- and rightward components

$$A(t, x) = A_+(x + t) + A_-(x - t) \tag{4.1}$$

so that the left- and right-movers possess the following Poisson brackets:

$$\{A_\pm(x), A_\pm(y)\} = \pm\tfrac{1}{4}\,\mathrm{sign}\,(x - y) \qquad \{A_+(x), A_-(y)\} = \mathrm{const}. \tag{4.2}$$

The free field bracket (3.7) is a consequence of these, but not vice versa. The constant in (4.2) is not assumed to be absolute (non-dynamical), i.e. it is not necessarily a central element of the Poisson structure; however, a more traditional choice for it is zero.

Let us discuss the splitting problem (4.1), (4.2) for the non-singular in-field described in the previous sections (the label 'in' will be omitted). In section 3 we agreed that the admissible algebra $\mathcal{O}'' \times \mathcal{O}''$ was sufficient for satisfactory Hamiltonian description of the in-field, in particular, we managed to reproduce the correct free-field bracket (3.7). A striking result is that, for the algebra $\mathcal{O}'' \times \mathcal{O}''$, the splitting (4.1), (4.2) is only possible if $bc = 0$, and all solutions have the following form.

If $b > 0$, $c = 0$, then

$$\begin{aligned} A_+(x) &= -\,\log|\chi_+(x)| + G(Q) \\ A_-(x) &= -\,\log|(1,0)T\Omega_-(x)| - G(Q) \end{aligned} \qquad Q = 2\log\left|\frac{a}{b}\right|.$$

Throughout this section G denotes an arbitrary smooth non-dynamical function of the dynamical variable(s) indicated.

If $b = 0$, $c > 0$, then

$$\begin{aligned} A_+(x) &= -\log\left|\Omega_+(x)T\begin{pmatrix}1\\0\end{pmatrix}\right| + G(Q) \\ A_-(x) &= -\log|\psi_-(x)| - G(Q) \end{aligned} \qquad Q = 2\log\left|\frac{a}{c}\right|.$$

If $b = 0$, $c = 0$, then

$$A_+(x) = -\log|\chi_+(x)| - \tfrac{1}{2}\log|a| + G$$

$$A_-(x) = -\log|\psi_-(x)| - \tfrac{1}{2}\log|a| - G.$$

Recall that in this case a, as well as G, is an absolute constant.

Let, now, the admissible algebra be $\mathcal{O}' \times \mathcal{O}'$. In this case the splitting (4.1), (4.2) is always possible, but the extent to which it is non-unique becomes greater. In what follows we shall be using the following combinations of the parameters participating in the definition of the Poisson bracket (3.2)–(3.6):

$$j = h_1 + h_2 - p + v \qquad l = k_1 + k_2 - u + q$$

$$m = h_1 - h_2 - p - v \qquad n = k_1 - k_2 - u - q.$$

For the admissible algebra $\mathcal{O}' \times \mathcal{O}'$, all solutions of the splitting problem (4.1), (4.2) have the following form.

If $b > 0$, $c > 0$, then

$$A_+(x) = -\log\left|\Omega_+(x)T\begin{pmatrix}1\\0\end{pmatrix}\right| + \tfrac{1}{2}\log|a| + \tfrac{1}{4}\frac{jm+ln}{m^2+n^2}\log|bc|$$

$$-\tfrac{1}{8}\frac{1}{m^2+n^2}(m\log|\beta_+| + n\log|\beta_-|) + G(N_1, N_2, Q)$$

$$A_-(x) = -\log|(1,0)T\Omega_-(x)| + \tfrac{1}{2}\log|a| - \tfrac{1}{4}\frac{jm+ln}{m^2+n^2}\log|bc|$$

$$+\tfrac{1}{8}\frac{1}{m^2+n^2}(m\log|\beta_+| + n\log|\beta_-|) - G(N_1, N_2, Q)$$

$$N_1 = -n\log|\beta_+| + m\log|\beta_-| + 2(jn - lm)\log|bc|$$

$$N_2 = \log\left|\frac{b}{c}\right| \qquad Q = -\log\left|\frac{bc}{a^2}\right|.$$

Note that $m^2 + n^2 \neq 0$ if the parameters of the Poisson bracket are chosen as in [7] (when $m^2 + n^2 = 0$ the splitting we are looking for does not exist).

If $b > 0$, $c = 0$, then

$$A_+(x) = -\log|\chi_+(x)| + G(N_1, N_2, Q)$$

$$A_-(x) = -\log|(1,0)T\Omega_-(x)| - G(N_1, N_2, Q) \qquad Q = 2\log\left|\frac{a}{b}\right|.$$

$$N_1 = \log|\beta_+| + 4m\log|a| - 4j\log|b|$$

$$N_2 = \log|\beta_-| + 4n\log|a| - 4l\log|b|$$

If $b = 0$, $c > 0$, then

$$A_+(x) = -\log\left|\Omega_+(x)T\begin{pmatrix}1\\0\end{pmatrix}\right| + G(N_1, N_2, Q)$$

$$A_-(x) = -\log|\psi_-(x)| - G(N_1, N_2, Q) \qquad Q = 2\log\left|\frac{a}{c}\right|.$$

$$N_1 = \log|\beta_+| - 4m\log|a| - 4j\log|c|$$

$$N_2 = \log|\beta_-| - 4n\log|a| - 4l\log|c|$$

If $b=0$, $c=0$, then

$$A_+(x) = -\log|\chi_+(x)| - \tfrac{1}{2}\log|a| + G(N_1, N_2, Q)$$

$$A_-(x) = -\log|\psi_-(x)| - \tfrac{1}{2}\log|a| - G(N_1, N_2, Q)$$

$$N_1 = \log|a| \qquad N_2 = l\log|\beta_+| - j\log|\beta_-|$$

$$Q = -\tfrac{1}{2}\frac{1}{j^2+l^2}(j\log|\beta_+| + l\log|\beta_-|).$$

Note that $j^2+l^2 \neq 0$ if the parameters of the Poisson bracket are chosen as in [7] (when $j^2+l^2=0$ the splitting also exists, but we do not describe it here).

In all cases considered, the function G must not depend on Q if one wants the constant in (4.2) to be zero. Also, in all cases the functionals N_1 and N_2 commute with the free field, and Q generates shifts of it, i.e.

$$\{N_1, A(t,x)\} = 0 \qquad \{N_2, A(t,x)\} = 0 \qquad \{Q, A(t,x)\} = 1.$$

Moreover, N_1 and N_2 are the only independent functionals that commute with the fundamental field Φ:

$$\{N_1, \Phi(t,x)\} = 0 \qquad \{N_2, \Phi(t,x)\} = 0.$$

Being a purely infinite-dimensional effect, this by no means contradicts either non-degeneracy of the bracket or the fundamental character of the field Φ.

Appendix A

Boundary behaviour of $A_{\rm in}$.

(1) $c \neq 0,\ a \neq 0 \Longleftrightarrow$

$$A_{\rm in}(t,x) = -\log(B^+_{\rm in}|(x-q^+_{\rm in})^2 - (t-\tau^+_{\rm in})^2|) + s_+(x)$$

$$q^+_{\rm in} + \tau^+_{\rm in} = -\frac{(-1)^{n_+}\alpha_+ + a/c}{|\beta_+|} \qquad q^+_{\rm in} - \tau^+_{\rm in} = -\frac{(-1)^{n_-}\alpha_- + b/a}{|\beta_-|}$$

$$B^+_{\rm in} = |c||\beta_+\beta_-|^{1/2}.$$

(2) $c = 0 \Longleftrightarrow$

$$A_{\rm in}(t,x) = -\log(B^+_{\rm in}|x-t-q^+_{\rm in}|) + s_+(x)$$

$$B^+_{\rm in} = |a|\left|\frac{\beta_-}{\beta_+}\right|^{1/2} \qquad q^+_{\rm in} = -\frac{(-1)^{n_-}\alpha_- + b/a}{|\beta_-|}.$$

(3) $b \neq 0,\ a \neq 0 \Longleftrightarrow$

$$A_{\rm in}(t,x) = -\log(B^-_{\rm in}|(x-q^-_{\rm in})^2 - (t-\tau^-_{\rm in})^2|) + s_-(x)$$

$$q^-_{\rm in} + \tau^-_{\rm in} = \frac{(-1)^{n_+}\vartheta_+ + c/a}{|\beta_+|} \qquad q^-_{\rm in} - \tau^-_{\rm in} = \frac{(-1)^{n_-}\vartheta_- + a/b}{|\beta_-|}$$

$$B^-_{\rm in} = |b||\beta_+\beta_-|^{1/2}.$$

(4) $b = 0 \Longleftrightarrow$

$$A_{\rm in}(t,x) = -\log(B_{\rm in}^{-}|x+t-q_{\rm in}^{-}|) + s_{-}(x)$$

$$B_{\rm in}^{-} = |a|\left|\frac{\beta_+}{\beta_-}\right|^{1/2} \qquad q_{\rm in}^{-} = \frac{(-1)^{n_+}\vartheta_+ + c/a}{|\beta_+|}.$$

(5) $a = 0 \Longleftrightarrow$

$$A_{\rm in}(t,x) = -\log(B_{\rm in}^{\pm}|x \pm t - q_{\rm in}^{\pm}|) - \log\frac{2|x - v_{\rm in}t - q_{\rm in}|}{\sqrt{1-v_{\rm in}^2}} + s_{\pm}(x)$$

$$B_{\rm in}^{\pm} = |\beta_{\pm}| \qquad q_{\rm in}^{+} = -\frac{\alpha_+}{\beta_+} \qquad q_{\rm in}^{-} = \frac{\vartheta_-}{\beta_-}$$

($v_{\rm in}$ and $q_{\rm in}$ are given by (2.12)).

Boundary behaviour of $A_{\rm out}$.

(1) $c \neq 0,\ d \neq 0 \Longleftrightarrow$

$$A_{\rm out}(t,x) = -\log(B_{\rm out}^{+}|(x-q_{\rm out}^{+})^2 - (t-\tau_{\rm out}^{+})^2|) + s_{+}(x)$$

$$q_{\rm out}^{+} + \tau_{\rm out}^{+} = -\frac{(-1)^{n_+}\alpha_+ + b/d}{|\beta_+|} \qquad q_{\rm out}^{+} - \tau_{\rm out}^{+} = -\frac{(-1)^{n_-}\alpha_- + d/c}{|\beta_-|}$$

$$B_{\rm out}^{+} = |c||\beta_+\beta_-|^{1/2}.$$

(2) $c = 0 \Longleftrightarrow$

$$A_{\rm out}(t,x) = -\log(B_{\rm out}^{+}|x+t-q_{\rm out}^{+}|) + s_{+}(x)$$

$$B_{\rm out}^{+} = |d|\left|\frac{\beta_+}{\beta_-}\right|^{1/2} \qquad q_{\rm out}^{+} = -\frac{(-1)^{n_+}\alpha_+ + b/d}{|\beta_+|}.$$

(3) $b \neq 0,\ d \neq 0 \Longleftrightarrow$

$$A_{\rm out}(t,x) = -\log(B_{\rm out}^{-}|(x-q_{\rm out}^{-})^2 - (t-\tau_{\rm out}^{-})^2|) + s_{-}(x)$$

$$q_{\rm out}^{-} + \tau_{\rm out}^{-} = \frac{(-1)^{n_+}\vartheta_+ + d/b}{|\beta_+|} \qquad q_{\rm out}^{-} - \tau_{\rm out}^{-} = \frac{(-1)^{n_-}\vartheta_- + c/d}{|\beta_-|}$$

$$B_{\rm out}^{-} = |b||\beta_+\beta_-|^{1/2}.$$

(4) $b = 0 \Longleftrightarrow$

$$A_{\rm out}(t,x) = -\log(B_{\rm out}^{-}|x-t-q_{\rm out}^{-}|) + s_{-}(x)$$

$$B_{\rm out}^{-} = |d|\left|\frac{\beta_-}{\beta_+}\right|^{1/2} \qquad q_{\rm out}^{-} = \frac{(-1)^{n_-}\vartheta_- + c/d}{|\beta_-|}.$$

(5) $d = 0 \Longleftrightarrow$

$$A_{\rm out}(t,x) = -\log(B_{\rm out}^{\pm}|x \mp t - q_{\rm out}^{\pm}|) - \log\frac{2|x - v_{\rm out}t - q_{\rm out}|}{\sqrt{1-v_{\rm out}^2}} + s_{\pm}(x)$$

$$B_{\rm out}^{\pm} = |\beta_{\mp}| \qquad q_{\rm out}^{+} = -\frac{\alpha_-}{\beta_-} \qquad q_{\rm out}^{-} = \frac{\vartheta_+}{\beta_+}$$

($v_{\rm out}$ and $q_{\rm out}$ are given by (2.13)).

Appendix B

The zero-time in-fields defined in section 2 may have a finite number of singular points. They may be of the following types only:

(1)

$$\varphi_{\text{in}}(x) = -\log|x-q| + \tilde{\varphi}(x) \qquad \pi_{\text{in}}(x) = -\frac{1}{x-q} - \tilde{\varphi}'(x) + \tilde{\pi}(x)$$

where $\tilde{\pi}, \tilde{\varphi}$ are C^∞-smooth in a neighbourhood of q, and $\tilde{\pi}(q) = 0$. The singular point $x = q$ generates a lightlike singular line of A_{in}, sloping to the left (it is assumed that the t axis is directed upwards and the x axis rightwards).

(2)

$$\varphi_{\text{in}}(x) = -\log|x-q| + \tilde{\varphi}(x) \qquad \pi_{\text{in}}(x) = \frac{1}{x-q} + \tilde{\varphi}'(x) + \tilde{\pi}(x)$$

where $\tilde{\pi}$, $\tilde{\varphi}$ are C^∞-smooth in a neighbourhood of q, and $\tilde{\pi}(q) = 0$. The singular point $x = q$ generates a lightlike singular line of A_{in}, sloping to the right.

(3)

$$\varphi_{\text{in}}(x) = -2\log|x-q| + \tilde{\varphi}(x) \qquad \tilde{\varphi}'(q) = 0 \qquad \pi_{\text{in}}(q) = 0$$

where $\tilde{\varphi}$, as well as π_{in}, is C^∞-smooth in a neighbourhood of q. The singular point of this type generates two lightlike singular lines of A_{in}, which intersect in the point $x = q$ at the instant $t = 0$.

(4)

$$\varphi_{\text{in}}(x) = \ldots - \log\frac{2|x-q_{\text{in}}|}{\sqrt{1-v_{\text{in}}^2}} \qquad \pi_{\text{in}}(x) = \ldots + \frac{v_{\text{in}}}{x-q_{\text{in}}}$$

where the dots stand for either some arbitrary smooth (in a neighbourhood of q_{in}) functions or one of the preceding formulae for the singular behaviour of φ_{in} and π_{in} (with q replaced by q_{in}). There may be only one singular point of this type, and if it is actually present, then the in-field must have the boundary behaviour of type 5 (see appendix A) with the same q_{in} and v_{in}. The singular point of this type generates a timelike singular line of A_{in} characterized by the velocity v_{in}. It also generates lightlike singular lines if the dots in the above formulae represent non-smooth functions.

The out-fields may have a finite number of singular points of the same types.

Tables B1 and B2 show the number of singular lines of A_{in} and A_{out} as a function of T. It is assumed that $U_\pm \in M_{n_\pm}$.

Table B1.

	The number of singular lines of A_{in}			
T	Lightlike sloping to the left (\)	Lightlike sloping to the right (/)	Timelike	Total
$a = 0$	n_+	n_-	1	$n_+ + n_- + 1$
$a > 0, b \geqslant 0, c \geqslant 0$	n_+	n_-	0	$n_+ + n_-$
$a > 0, b \geqslant 0, c < 0$	$n_+ + 1$	n_-	0	$n_+ + n_- + 1$
$a > 0, b < 0, c \geqslant 0$	n_+	$n_- + 1$	0	$n_+ + n_- + 1$
$a > 0, b < 0, c < 0$	$n_+ + 1$	$n_- + 1$	0	$n_+ + n_- + 2$

Table B2.

T	The number of singular lines of A_{out} Lightlike sloping to the left (\)	Lightlike sloping to the right (/)	Timelike	Total
$d=0$	n_+	n_-	1	$n_+ + n_- + 1$
$d>0, b\geqslant 0, c\geqslant 0$	n_+	n_-	0	$n_+ + n_-$
$d>0, b\geqslant 0, c<0$	n_+	$n_- + 1$	0	$n_+ + n_- + 1$
$d>0, b<0, c\geqslant 0$	$n_+ + 1$	n_-	0	$n_+ + n_- + 1$
$d>0, b<0, c<0$	$n_+ + 1$	$n_- + 1$	0	$n_+ + n_- + 2$

Appendix C

The ingredients of A_{in} (when $a=0$) and A_{out} (when $d=0$) are transformed under conformal transformations as follows:

$$\psi^F_{\pm 1}(x) = \psi_{\pm 1}(F^\pm(x))\left(\frac{\partial F^\pm(-)}{\partial F^\pm(x)}\right)^{1/2}$$

$$\chi^F_{\pm 1}(x) = \chi_{\pm 1}(F^\pm(x))\left(\frac{\partial F^\pm(+)}{\partial F^\pm(x)}\right)^{1/2}$$

$$v^F_{\text{in}} = \frac{v_{\text{in}} - \nu}{1 - v_{\text{in}}\nu} \qquad \nu = \frac{A^+_- - A^-_+}{A^+_- + A^-_+}$$

$$q^F_{\text{in}} = \frac{2q_{\text{in}} - B^+_- - B^-_+ + v_{\text{in}}(B^+_- - B^-_+)}{A^+_- + A^-_+ - v_{\text{in}}(A^+_- - A^-_+)}$$

$$v^F_{\text{out}} = \frac{v_{\text{out}} - \mu}{1 - v_{\text{out}}\mu} \qquad \mu = \frac{A^+_+ - A^-_-}{A^+_+ + A^-_-}$$

$$q^F_{\text{out}} = \frac{2q_{\text{out}} - B^+_+ - B^-_- + v_{\text{out}}(B^+_+ - B^-_-)}{A^+_+ + A^-_- - v_{\text{out}}(A^+_+ - A^-_-)}$$

where As and Bs are defined through F as follows:

$$F^+(x) = A^+_\pm x + B^+_\pm + s_\pm(x) \qquad F^-(x) = A^-_\pm x + B^-_\pm + s_\pm(x).$$

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